Arguments of zeros of highly log concave polynomials

David Handelman[*]

Abstract. For a real polynomial $p = \sum_{i=0}^{n} c_i x^i$ with no negative coefficients and $n \geq 6$, let $\beta(p) = \inf_{i=1}^{n-1} c_i^2/c_{i+1}c_{i-1}$ (so $\beta(p) \geq 1$ entails that p is log concave). If $\beta(p) > 1.45\ldots$, then all roots of p are in the left half plane, and moreover, there is a function $\beta_0(\theta)$ (for $\pi/2 \leq \theta \leq \pi$) such that $\beta \geq \beta_0(\theta)$ entails all roots of p have arguments in the sector $|\arg z| \geq \theta$ with the smallest possible θ; we determine exactly what this function (and its inverse) is (it turns out to be piecewise smooth, and quite tractible). This is a one-parameter extension of Kurtz's theorem (which asserts that $\beta \geq 4$ entails all roots are real). We also prove a version of Kurtz's theorem with real (not necessarily nonnegative) coefficients.
MSC2010: 26C10, 30C15, 05E99

As an outgrowth of a question concerning a class of analytic functions, we give criteria for all roots of real polynomials to lie in a sector of the form $\{z \in \mathbf{C} \mid |\arg z| > \theta\}$, at least for $\pi \geq \theta \geq \pi/2$ and asymptotically as $\theta \to 0$. The criteria depend only on log concavity of the coefficients.

Specifically, if $f = \sum_{i=0}^{N} c_i x^i$ (of degree $N \geq 6$) is a polynomial with positive coefficients, let $\beta := \inf_{i=1}^{N-1} c_i^2/c_{i+1}c_{i-1}$, and assume $\beta > 1$. Then there is $\theta > 0$ such that for all roots, z, of f, $|\arg z| > \theta$ (where arg is the principal value, i.e., arg takes on values in $(-\pi, \pi]$). The function $\beta \mapsto \theta$ is determined exactly for $\pi/2 \leq \theta \leq \pi$. For example, if $\beta = 1 + \sqrt{2}$, then all roots of f lie in the sector $|\arg z| > 3\pi/4$, while if $\beta = 2$, then all roots lie in $|\arg z| > 2\pi/3$, and moreover, these numbers are sharp.

We also show that if the c_i are assumed merely to be complex, then if $\beta := \inf |c_i|^2/|c_{i+1}c_{i-1}| \geq 4.45\ldots$ (a root of a transcendental equation), then f has only simple roots and can be located within specific annuli), and moreover, if the c_i are real, then all roots of f are real. This is an extension of Kurtz's theorem, which states that if the c_j are all positive and $\beta > 4$, then all roots are real. We also provide minor improvements on this result.

Then we consider in section 2 an old question [P] and [CC, section 4] (I am indebted to Tom Craven for these references). Form the entire function (or the polynomial) $g_\beta = \sum c_i x^i$ wherein the quotients $\beta := c_i^2/c_{i+1}c_{i-1}$ do not change in i. For what values of β does g_β have only real roots? We provide an answer, $\beta \geq \beta_0$, with β_0 determined to 24 places (and show how to improve this), but unfortunately β_0 does not appear to be connected to anything else. However, it does yield apparent paradoxes; for example, there exists a polynomial (of any degree exceeding 5) g for which $\beta(g) > 3.99$, but which has nonreal roots; however, since $\beta_0 < 3.3$, any g_β with $\beta > 3.3$ will have only real roots.

The original question that led to this article, was the determination of conditions on a polynomial g guaranteeing all roots lie in the sector $|\arg z - \pi| < \pi/4$, and by the result cited above, $\beta \geq 1 + \sqrt{2}$ (in the presence of $N \geq 5$) is sufficient. This question itself emanated from a result in [H], guaranteeing that a polynomial with $g(1) = 1$ belong to a class of analytic functions known there as $\mathcal{E}$, which play a role in classification criteria for AT ergodic transformations.

Section 1 Arguments of zeros

Here we give sufficient—but far from necessary—conditions for polynomials and entire functions to have all their roots in this sector, which however, are easy to verify. However, we have more

* Supported by an NSERC Discovery Grant

precise results for sectors of the form $\{z \in \mathbf{C} \mid |\arg z| > \theta\}$ for all θ with $\pi > \theta \geq \pi/2$.

A well-known theorem due to Kurtz [K] asserts that if $p = \sum_{i=0}^{N} c_i z^i$ is a polynomial of degree N with $c_0 > 0$ and only nonnegative coefficients such that for all $i = 1, \ldots, N-1$, the numbers $\beta_i(p) := c_i^2/c_{i+1}c_{i-1}$ all exceed 4, then all roots of p are real (and thus negative). This is extended to give similar type conditions (on the ratios, $\beta_i(p)$) to guarantee that all the zeros lie in a sector of the form $|\arg z - \pi| < \psi$ for $0 < \psi \leq \pi/2$. As a special case, we show that if $N \geq 5$ and $c_i^2/c_{i+1}c_{i-1} \geq 1 + \sqrt{2}$, then all zeros satisfy $|\arg z - \pi| < \pi/4$, so that the corresponding $p/p(1)$ belongs to $\mathcal{E}$. The number $1 + \sqrt{2}$ is sharp in the sense that for all $\epsilon > 0$, there exists a polynomial, p, of degree N with some roots outside the sector, yet with $c_i^2/c_{i+1}c_{i-1} > 1 + \sqrt{2} - \epsilon$ for all $i = 1, \ldots, N-1$.

For $\theta = \pi - \psi$, define two functions,

$$\mathcal{R}(\theta) = \begin{cases} \text{largest positive real root of } X^2 - 2\cos\theta \cdot X^{3/2} + 2\cos 2\theta = 0 & \text{if one exists} \\ 1 & \text{otherwise} \end{cases}$$

$$\mathcal{S}(\theta) = \begin{cases} \text{largest positive real root of } X^3 + \frac{\cos 3\theta/2}{\cos\theta/2} \cdot X^2 + \frac{\cos 5\theta/2}{\cos\theta/2} = 0 & \text{if one exists} \\ 1 & \text{otherwise.} \end{cases}$$

We will show that if $N \geq 6$ (or $N \geq 5$ if θ is not too close to $\pi/2$), $\pi/2 \leq \theta < \pi$ and

$$\inf_{i=1}^{N-1} \frac{c_i^2}{c_{i+1}c_{i-1}} \geq \max\left\{4\cos^2\theta, 1 - 2\cos\theta, \mathcal{R}(\theta), \mathcal{S}(\theta)\right\},$$

then all roots of $p = \sum c_i z^i$ lie in the sector $|\arg z| < \theta$, and moreover, this is sharp (in the sense of the example with $\theta = 3\pi/4$ and $1 - 2\cos\theta = 1 + \sqrt{2}$); this criterion can be slightly simplified, as in the statement of Theorem 1.6.

For example, with $\theta = \pi/2$, all roots of p lie in left half plane if $\inf_i c_i^2/c_{i+1}c_{i-1}$ is at least as large as the positive real root of $X^3 - X^2 - 1 = 0$, about $1.45\ldots$. For $\theta = 2\pi/3$, the corresponding lower bound is 2.

These results extend in a very routine way to entire functions. When $g = \sum c_i z^i$ with $c_i > 0$ is entire and $\inf c_i^2/c_{i+1}c_{i-1} > 1$, then g is of order zero, and admits a factorization of the form $g/g(0) = \prod(1 - z/z_k)$ where z_k runs over the zeros.

Since we deal almost exclusively with polynomials all of whose zeros lie in $|\arg z| \geq \pi/2$, these polynomials will automatically have no negative coefficients, and if $|\arg z| \geq 2\pi/3$, the polynomials will be strongly unimodal (that is, the sequence consisting of their coefficients will be log concave). We give a result, Theorem 1.1, along the lines indicated here that does not require the coefficients to be nonnegative, but merely real, with the conclusion that the roots are all real. Of course, there is a vast literature on polynomials and entire functions all of whose zeros are real, but I couldn't find this particular result in the literature (which of course does not mean that it does not exist therein).

The lower bound given in Theorem 1.1, β_0, is likely not sharp, but on the other hand, as was noted in [K], whatever the optimal value is, it must be at least $25/6 > 4$, because the polynomial $x^3 - 5x^2 + 6x + 1$ has nonreal roots.

Define the function $F(r, \beta) = 1 + r + r^2/\beta + r^3/\beta^3 + \cdots = \sum_{j=0}^{\infty} r^j/\beta^{j(j-1)/2}$; typically, $\beta > 2$ and $0 \leq r \leq 1$. It is perhaps accidental that the function $F_\beta(z) = F(z, \beta)$ is an entire function (when $\beta > 1$) satisfying $\beta_k = \beta^2$ (so if $\beta \geq 2$, then all of its roots are real).

Suppose that $\{D_k\}$ is a summable sequence of positive real numbers and $\beta_j := D_j^2/D_{j+1}D_{j-1} \geq \beta$ for some number $\beta > 1$. Since the sequence is strongly unimodal, it is unimodal, and let m be

the smallest mode, and let $k \geq m$, so that the sequence $\{D_j\}_{j \geq k}$ is monotone nonincreasing. Set $r = D_{k+1}/D_k$; necessarily, $r \leq 1$. For $l > k+1$,

$$\begin{aligned} \frac{D_l}{D_{l+1}} &= \frac{D_{l-1}}{D_l}\frac{1}{\beta_{l-1}}, \qquad \text{which iterates to} \\ &= \frac{D_{k+1}}{D_k}\frac{1}{\beta_{l-1}\beta_{l-2}\dots\beta_{k+1}} = \frac{r}{\prod_{j=k+1}^{l-1}\beta_j}. \end{aligned}$$

If we also assume that $\beta_j \geq \beta$ for all j, then we have $D_l/D_{l+1} \leq r\beta^{l-k-1}$. Now D_l/D_k telescopes as a product $(D_l/D_{l-1})(D_{l-1})/D_{l-2}\cdots$, and we thus obtain that $D_l \leq r^{l-k-1}\beta^{(l-k-1)(l-k-2)}D_k$. Hence the mass of the tail, that is, the sum $\sum_{j \geq k} D_j$ is bounded above by $F(r,\beta)D_k$.

Similarly, if $k \leq m$, the sequence is increasing, and on setting $s = D_{k-1}/D_k$, we have $\sum_{j \leq k} D_j < F(s,\beta)D_k$; the inequality is strict, because the sequence is finite to the left of m.

Now we are in position to prove a result via an easy application of Rouch's theorem, where we use a single monomial as the function to which we compare the zeros. To avoid cluttering the statement of the theorem even more than it currently is, we define

$$\begin{aligned} \rho_k &:= \max\left\{\frac{d_{k-2}}{rd_{k-1}}, \frac{d_k}{2d_{k+1}F(r,\beta)}\left(1 - \sqrt{1 - \frac{4F(r,\beta)}{\beta_k}}\right)\right\} \\ R_k &:= \min\left\{\frac{rd_{k+1}}{d_{k+2}}, \frac{d_k}{2d_{k+1}F(r,\beta)}\left(1 + \sqrt{1 - \frac{4F(r,\beta)}{\beta_k}}\right)\right\}, \end{aligned}$$

where $r = \beta^{-3/2}$.

THEOREM 1.1 Let $f = \sum_{j=0}^{N} c_j z^j$ (with complex c_j) be an entire function with N in $\{3, 4, \dots\} \cup \{\infty\}$. Suppose that the sequence $(d_j = |c_j|)$ satisfies $\beta_j := d_j^2/d_{j+1}d_{j-1} \geq \beta_0$ for all j, where $\beta_0 = 4.448505576\dots$ is the unique root of $F(\beta^{-3/2}, \beta)^2 = \beta$. Then all zeros of f are simple, and exactly one appears in the annulus $R_k < |z| < \rho_k$, and there are no others. If additionally, c_j are all real, then all zeros of f are real.

Proof. We will apply Rouch's theorem with $g(z) = c_k z^k$ (once for each k); we will show that $|(f-g)(z)| < |g(z)|$ on the circle $|z| = R$, with R to be chosen appropriately. It follows that f has exactly k zeros in the disk, and by increasing k to $k+1$, we will obtain a larger disk containing exactly $k+1$ zeros, so there must be exactly one zero in the set-theoretic difference of the disks, that is, an annulus.

For unspecified $R > 0$, set $D_j = d_j R^j = |c_j z^j|$ (on $|z| = R$). Fix k, and suppose that $d_k \geq d_{k+1}R, d_{k-1}/R$; that is, k is a mode of the sequence $\{d_j R^j\}$. Define $r = \max\{D_{k+2}/D_{k+1}, D_{k-2}/D_{k-1}\}$, so that $r \leq 1$. Set $\beta = \inf\{\beta_j\}$. Then we have

$$\begin{aligned} \sum_{j \geq 1} D_{k+j} &\leq D_{k+1}F(r,\beta) \\ \sum_{j \geq 1} D_{k-j} &< D_{k-1}F(r,\beta). \end{aligned}$$

Obviously, for z on the circle, $|g(z)| = d_k R^k = D_k$; equally obviously, $|(f-g)(z)| \leq \sum_{j \neq k} D_j$, so to show $|(f-g)(z)| < |g(z)|$, it would suffice to show that $D_k \geq F(r,\beta)(D_{k+1} + D_{k-1})$. We now

derive conditions on R, r, and β to guarantee this; it is equivalent to the quadratic inequality,

$$R^2 d_{k+1} - \frac{R d_k}{F(r,\beta)} + d_{k-1} \leq 0 \quad \text{that is,}$$
$$\begin{cases} \beta_k \geq 4F(r,\beta) \text{ and} \\ \frac{d_k}{2d_{k+1}F(r,\beta)} \left(1 - \sqrt{1 - \frac{4F(r,\beta)}{\beta_k}}\right) \leq R \leq \frac{d_k}{2d_{k+1}F(r,\beta)} \left(1 + \sqrt{1 - \frac{4F(r,\beta)}{\beta_k}}\right). \end{cases}$$

To summarize, at this point, we require conditions on R (which is to be determined), and r (also to be determined), as follows:

$$\frac{d_{k-1}}{d_k} \leq R \leq \frac{d_k}{d_{k+1}}$$
$$\frac{d_k}{2d_{k+1}F(r\beta)} \left(1 - \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right) \leq R \leq \frac{d_k}{2d_{k+1}F(r\beta)} \left(1 + \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right)$$
$$\frac{D_{k+2}}{D_{k+1}}, \frac{D_{k-2}}{D_{k-1}} \leq r,$$

and to make r as small as possible subject to these conditions. The third condition is equivalent to

$$(*) \qquad \frac{d_{k-2}}{r d_{k-1}} \leq R \leq \frac{r d_{k+1}}{d_{k+2}},$$

which actually subsumes the first. Necessary and sufficient for the existence of $R > 0$ satisfying (*) is simply $d_{k+1}d_{k-1}/d_{k+2}d_{k-2} \geq 1/r^2$, that is, $\beta_{k-1}\beta_k\beta_{k+1} \geq 1/r^2$. (The left side will be at least 64, so this allows r at this stage to be fairly small.) For the second condition and (*) to be compatible, we require the two inequalities (which now constrain r).

$$\frac{d_{k-2}}{r d_{k-1}} \leq \frac{d_k}{2d_{k+1}F(r,\beta)} \left(1 + \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right) \quad \text{and}$$
$$\frac{d_k}{2d_{k+1}F(r,\beta)} \left(1 - \sqrt{1 - \frac{4F(r,\beta_{\beta_k})}{\beta_k}}\right) \leq \frac{r d_{k+1}}{d_{k+2}}.$$

These two inequalities can be written in the form

$$\frac{1}{r} \leq \frac{\beta_k \beta_{k-1}}{2F(r,\beta_k)} \left(1 + \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right)$$
$$\frac{1}{r} \leq \frac{2\beta_k F(r,\beta_k)}{1 - \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}} = \frac{2\beta_k F(r,\beta_k)}{4F(r,\beta_k)/\beta_k} \left(1 + \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right)$$
$$= \frac{\beta_k^2 F(r,\beta_k)}{2} \left(1 + \sqrt{1 - \frac{4F(r,\beta_k)}{\beta_k}}\right)$$

Now suppose that $\beta \leq \beta_k, \beta_{k\pm1}$. Since $F(r,\beta) > 1$ (when $r > 0$), sufficient for both these inequalities to hold is

$$\frac{1}{r} \leq \frac{\beta^2}{2F(r,\beta)} \left(1 + \sqrt{1 - \frac{4F(r,\beta)}{\beta}}\right).$$

Therefore, it is sufficient to find r and β so that (for suitable β), the following hold:

$$\begin{aligned} 4F(r,\beta) &\leq \beta \\ \frac{1}{r} &\leq \min\left\{\beta^{3/2}, \frac{\beta^2 F(r,\beta)}{2}\left(1+\sqrt{1-\frac{4F(r,\beta)}{\beta}}\right)\right\} \end{aligned}$$

Normally, this would be hopeless; however, there is a trick, obtained by setting the two terms in the minimum to each other. Let β_0 be the unique positive solution to $\beta = (F(\beta^{-3/2},\beta)+1)^2$; a back of the envelope calculation yields easily that $4.3 < \beta_0 < 4.5$. *Maple* yields $4.448505576\ldots$ (convergence is extremely fast). Setting $r = \beta_0^{-3/2}$, we see fairly quickly that all the relevant inequalities hold.

Hence if each of $\beta_k, \beta_{k\pm 1}$ are at least as large as β_0, then Rouch's theorem applies, and we deduce that with

$$\max\left\{\frac{d_{k-2}}{rd_{k-1}}, \frac{d_k}{2d_{k+1}F(r,\beta)}\left(1-\sqrt{1-\frac{4F(r,\beta)}{\beta_k}}\right)\right\} \leq R \leq$$
$$\leq \min\left\{\frac{rd_{k+1}}{d_{k+2}}, \frac{d_k}{2d_{k+1}F(r,\beta)}\left(1+\sqrt{1-\frac{4F(r,\beta)}{\beta_k}}\right)\right\},$$

the interval is nonempty (it could be a singleton), and f has exactly k zeros in $|z| < R$ and none on $|z| = R$. Then ρ_k is the left endpoint and R_k is the right endpoint. Assume that $\beta_j \geq \beta_0$ for all j. Necessarily $\rho_{k+1} > R_k$. It follows that on the annulus $\rho_k \leq |z| \leq R_k$, f has no zeros, and on the annulus $R_k < |z| < \rho_{k+1}$, f has exactly one root. In particular, all roots are simple. Moreover, if we additionally assume that all c_j are real, then the roots must be real (since nonreal roots come in conjugate pairs). •

Since the argument is based on Rouch's theorem, the constant is unlikely to be optimal.

Now we can slightly extend Kurtz's theorem [K]. By restricting to polynomials of higher degree (at least 3), we can replace the strict inequalities that appear in the original statement by greater than or equal signs. We also give rather crude ranges for the locations of the zeros.

For a nonzero real number r, define sign $(r) = -1$ if $r < 0$ and sign $(r) = 1$ if $r > 0$.

THEOREM 1.2 Suppose that $f = \sum_{j=0}^N c_j z^j$ with $N \in \{3,4,\ldots,\} \cup \{\infty\}$ is entire, all $c_j > 0$, and for all $j = 1, 2, \ldots, N-1$, we have $\beta_j := c_j^2/c_{j+1}c_{j-1} \geq 4$. Then all roots of f are simple and real, and moreover, if $-x_k$ is the kth smallest (negative) root, then

$$\frac{c_{k-1}}{2c_k}\left(1+\sqrt{1-\frac{4}{\beta_{k-1}}}\right) \leq x_k \leq \frac{c_k}{2c_{k+1}}\left(1-\sqrt{1-\frac{4}{\beta_k}}\right),$$

and this interval is nontrivial.

Proof. Define the closed interval

$$J_k = \left[\frac{c_k}{c_{k+1}}\left(1-\sqrt{1-\frac{4}{\beta_k}}\right), \frac{c_k}{c_{k+1}}\left(1+\sqrt{1-\frac{4}{\beta_k}}\right)\right].$$

This is a singleton when $\beta_k = 4$, otherwise it has nonzero length. We will show that for x in J_k, sign $(f(-x)) = (-1)^k$.

By strong unimodality of $\{c_j\}_{j=0}^N$, the sequence $\left\{x^j c_j\right\}$ is unimodal for any $x > 0$. Suppose positive x satisfies

$$(*) \qquad c_k x^k \geq c_{k+1} x^{k+1} + c_{k-1} x^{k-1};$$

then the sequence $\left\{x^j c_j\right\}$ has a maximum at $j = k$, hence is decreasing for $j > k$ and increasing for $j < k$. Since the expansion of $f(-x)$ is alternating, it follows immediately that $|f(-x)|$ is at least as large as $\sum_{j \leq k-2} c_j(-x)^j + \sum_{j \geq k+2} c_j(-x)^j$; at least one of these partial sums is not zero, since degree f is $N \geq 3$, and whenever one of the partial sums or $(-1)^k(c_k x^k - c_{k-1}x^{k-1} - c_{k+1}x^{k+1})$ is not zero, the sign is simply $(-1)^k$. Hence $f(-x)$ would have sign $(-1)^k$.

Now (*) is equivalent to the quadratic inequality $x^2 - xc_k/c_{k+1} + c_{k-1}/c_{k+1} \leq 0$; this in turn is equivalent to $(x - c_k/2c_{k+1})^2 \leq (c_k^2 - 4c_{k-1}c_{k+1})/4c_{k+1}^2$. This has a real solution if and only if $\beta_k \geq 4$, and in that case, the solutions are precisely the points of J_k.

For $k = 0$, the corresponding result is the obvious $f(-x) > 0$ if $0 < x \leq c_0/c_1$. Now it follows (easily) from strong unimodality of the sequence, that the right endpoint of J_k is less than the left endpoint of J_{k+1}. Hence f has at least $N + 1$ sign changes on the negative reals. Now suppose that N is finite. It has at least N distinct negative roots. Since the degree of f is N, this must account for all of them. Moreover, the roots can only occur between the J_k, that is, between the right endpoint of J_k and the left endpoint of J_{k+1}, which yields the range in the statement of the theorem.

Now suppose that $N = \infty$ and f is entire. Set $f_n = \sum_{j=0}^N c_j z^j$; then $f_n \to f$ uniformly on compact subsets of $\mathbf{C}$, and of course f is not identically zero. Let z_0 be a root of f. If z_0 is not on the negative real axis, then it has a neighbourhood which misses then negative reals. Thus none of the f_n have zeros on this neighbourhood, hence f cannot have a zero therein (since f is not identically zero), a contradiction. So all the real zeros of f lie on the negative reals, and we also know that the sign of $f(-x)$ does not change on any J_k. So all zeros lie in the indicated sets, and there is at least one (because of the sign changes) in each one. It remains to show there can be no more than one.

This is again a consequence of uniform convergence on compact sets; let D_k be the open disk centred at the midpoint between $-J_k$ and $-J_{k+1}$ and whose diameter joints the right endpoint of one to the left endpoint of the other. On the bounding circle, note that f_n'/f_n converges uniformly to f'/f, so the number of zeros enclosed also converges; hence f has just one zero in D_k. •

Now we want to obtain conditions on the ratios $(c_i^2/c_{i+1}c_{i-1})$ to guarantee that all the zeros are in sectors of the form $|\arg z| > \theta$ for given θ in $[\pi/2, \pi)$. Let P_N denote the collection of real polynomials of degree N or less, topologized by identifying the polynomial $\sum c_j z^j$ with the point (c_j) in $\mathbf{R}^{N+1}$.

LEMMA 1.3 Suppose that U is a nonempty open subset of P_N, and W is a nonempty regular open (equal to the interior of its closure) subset of $\mathbf{C}$ with the following properties.
(a) U is connected and all elements of U are degree N
(b) no element of U has a zero on ∂W, the boundary of W
(c) there exists p in U such that all of its zeros lie in W.
Then all zeros of all members of U lie in W.

Proof. Set $A = \{g \in U \mid \text{all zeros of } g \text{ lie in } W\}$; then A is nonempty. We show both A and $U \setminus A$ are open. For g in A, there exist tiny disks centred about its zeros, all of which, including their closure, are contained in W. On each such disk, we may assume that g does not vanish on the bounding circle; say δ is the infimum of the values of $|g|$ on the union of the circles. If h is in U and

$\|h-g\|$ (the norm is the absolute sum of the coefficients) is sufficiently small, then we can apply Rouch's theorem to h and g (on the union of the disks), and so deduce that h has N zeros within the union of the disks; since h has degree N, this accounts for all of its zeros. Hence h belongs to A.

The argument for $U \setminus A$ is similar, but we only have to work with one zero, z_0. If g is in $U \setminus A$ and z_0 is a zero of g not in W, then z_0 must lie in the complement of the closure of W (since W is regular, and h has no zeros on ∂W). Hence there exists a disk therein centred at z_0. The same Rouch's theorem argument (this time for a single disk) yields that any h sufficiently close to g (in the coefficientwise norm) must have a zero in the disk.

Since U is connected and A is open, $U = A$. •

Let $\mathbf{b} := (b_1, \ldots, b_{N-1})$ be a sequence of positive numbers, and define the following sets

$$U_0(\mathbf{b}) = \left\{ c = (c_j)_{j=0}^{N} \in (\mathbf{R}^{++})^{N+1} \;\middle|\; \beta_j(c) := \frac{c_j^2}{c_{j+1}c_{j-1}} > b_j,\ j = 1, 2, \ldots, N-1 \right\}$$
$$C(\mathbf{b}) = \left\{ (x_j)_{j=1}^{N-1} \in (\mathbf{R}^{++})^{N-1} \mid x_j > b_j,\ j = 1, 2, \ldots, N-1 \right\}.$$

Obviously $C(\mathbf{b})$ is $(b_j, \infty)^{N-1}$ and $U_0(\mathbf{b})$ is open in $\mathbf{R}^{N+1}$. Define $\phi \equiv \phi(\mathbf{b}) : U_0(\mathbf{b}) \to (\mathbf{R}^{++})^2 \times C(\mathbf{b})$ via $\phi((c_0, c_1, c_2, \ldots, c_N)) = (c_0, c_1, \beta_1(c), \beta_2(c), \ldots, \beta_{N-1}(c))$. This map is obviously well-defined and continuous. To construct its inverse, we note the recursive equations, $c_{j+1} = c_j^2/\beta_j c_{j-1}$. Iterating these yields each c_j ($j \geq 2$) as a function of c_0, c_1 and the β_j, and all the denominators that appear are strictly positive. It is immediate that this yields a (trivially) rational mapping (hence continuous) that is the inverse of ϕ. In particular, $U_0(\mathbf{b})$ is homeomorphic to $\mathbf{R}^{N+1}$, and is thus connected!

Let $U(\mathbf{b})$ be the image of $U_0(\mathbf{b})$ in P_N, that is, associate to c in U_0 the polynomial $f_c = \sum c_j z^j$. So we have that $U(\mathbf{b})$ is an open and connected subset of $P(\mathbf{b})$. Supppose $U(\mathbf{b})$ and W satisfy the conditions of the lemma. For each choice of $c_0, c_1 > 0$, take the closure of the points in U whose first two coordinates are (c_0, c_1); then take the union over all strictly positive choices of (c_0, c_1). The resulting set, call it $V(\mathbf{b})$ is easily described: it is the set of points $c = (c_j)$ for which the corresponding $\beta_j \geq b_i$.

Suppose that no members of $V(\mathbf{b})$ have zeros on ∂W; then it is easy to show that all zeros of all members of $V(\mathbf{b})$ lie in W (this is a bit surprising, since the latter is open). For if z_0 is a zero of f in $V(\mathbf{b})$, then z_0 cannot belong to ∂W by hypothesis, so if z_0 is not in W, it must lie in the complement of the closure of W, hence there is a disk centred at it that lies entirely in the complement of the closure. There exist f_n in $U(\mathbf{b})$ converging coordinatewise to f (with the first two coordinates fixed), hence $f_n \to f$ on compact sets, and once again, so f can have no zeros in the disk, a contradiction.

We state this as a corollary.

COROLLARY 1.4 *Let $\mathbf{b} = (b_1, \ldots, b_{N_1})$ be an N-tuple of real numbers with $b_i > 1$, and set*

$$V(\mathbf{b}) = \left\{ f = \sum_{j=0}^{N} c_j z^j \in P_N \;\middle|\; c_j > 0,\ \frac{c_j^2}{c_{j+1}c_{j-1}} \geq b_i \right\}.$$

Suppose that W is a regular open subset of $\mathbf{C}$ with boundary ∂W, and there exists f in $V(\mathbf{b})$ that has all of its zeros in W. Suppose that every f in $V(\mathbf{b})$ has no zeros in ∂W. Then all the zeros of every f in $V(\mathbf{b})$ lie in W.

For an angle $0 < \theta < \pi$, let W_θ denote the open sector in $\mathbf{C}$, $\{z \in \mathbf{C} \mid |\arg z| > \theta\}$ (the branch of $\arg z$ has values $-\pi < \arg z \leq \pi$), and let L_θ denote the ray in the upper half-plane,

$\{\lambda e^{i\theta} \mid \lambda \geq 0\}$. Then $\partial W_\theta = L_\theta \cup \overline{L_\theta}$. Since the polynomials in $U(\mathbf{b})$ are real, to verify condition (b) in Lemma 1.3, we need only verify that all polynomials therein have no zeros on L_θ.

Now to verify f has no zeros on L_θ for any f in U, we can make a further reduction. The reparameterization maps, $f \mapsto f_\lambda$ (for each $\lambda > 0$), where $f_\lambda(z) = f(\lambda z)$, do not change the β_j values. Hence f belongs to $U(\mathbf{b})$ if and only if every or any f_λ does. Thus if some f in U has a zero on L_θ, then there exists f_0 in U that vanishes at $e^{i\theta}$, that is, the point in L_θ on the unit circle. Thus it would suffice to show that $f(e^{i\theta})) \neq 0$ for all f in U. But we can do a bit better.

We claim that all the zeros of $V(\mathbf{b})$ also lie in W_θ. Suppose not; there exists $f = \sum c_j z^j$ in $V(\mathbf{b})$ with a zero, z_0, not in W_θ. There exist f_n whose first two coefficients are c_0 and c_1 respectively, with $f_n \to f$ coordinatewise. Since W_θ is regular, either z_0 is in ∂W or in the complement of $W \cup \partial W$; but the latter is impossible from $f_n \to f$.

Now let $\mathbf{b} = (\beta, \beta, \beta, \dots, \beta)$ for some $\beta > 1$, and let U_β (V_β) denote $U(\mathbf{b})$ ($V(\mathbf{b})$, respectively).

Let $f = \sum_{j=0}^{N} c_j z^j$, with $N \in \{5, 6, 7, \dots\} \cup \{\infty\}$ be an entire function with strictly positive coefficients such that for all $1 \leq j < N$, the numbers $\beta_j := c_j^2/c_{j+1}c_{j-1}$ are all at least as large as $\beta > 1$. We determine conditions on β (and to a lesser extent, on N), to guarantee that f does not vanish at $e^{i\theta}$, for suitable values of θ.

First assume that $N < \infty$, so we are dealing with polynomials.

We note that if f is replaced by its opposite (obtained by reversing the order of the coefficients), the set of β_j does not change (only the indexing), and any zeros on the unit circle that are zeros of the opposite function are zeros of the original. Hence in trying to show that $f(e^{i\theta}) \neq 0$ for all members of U_β, we can assume that if the mode appears at k (that is, $c_k \geq c_j$ for all j; since $\beta > 1$, there is at most one other mode, which must be adjacent), then $c_{k+1} \leq c_{k-1}$ (or vice versa, whichever works out better).

For a polynomial f, let $Z(f)$ denote its set of zeros (multiplicities are irrelevant for this discussion). For $0 < \theta < \pi$, let W_θ be $\{z \in \mathbf{C} \mid |\arg z - \pi| < \pi - \theta\}$, and denote $\pi - \theta$ by ψ. Define

$$\beta_0(\theta) = \inf \{\beta \mid f \in U_\beta \text{ implies } Z(f) \subset W_\theta\}.$$

(If $f \in U_4$ and $N \geq 5$, then $Z(f)$ consists of negative real numbers, hence belongs to W_θ; hence $\beta_0(\theta) \leq 4$.)

We can easily obtain lower bounds for $\beta_0(\theta)$ when $\pi/2 \leq \theta < \pi$; by more difficult methods, we show these are sharp for each $N \geq 6$. The functions $\mathcal{R}$ and $\mathcal{S}$ are defined in the introduction. Finally, we can state the main result of this section. For $N \geq 6$,

$$\beta_0(\theta) = \begin{cases} 4\cos^2\theta & \text{if } 4\pi/5 \leq \theta < \pi \\ 1 - 2\cos\theta & \text{if } \theta_0 \leq \theta \leq 4\pi/5 \\ \mathcal{R}(\theta) & \text{if } \theta_1 \leq \theta \leq \theta_0 \\ \mathcal{S}(\theta) & \text{if } \pi/2 \leq \theta \leq \theta_1 \end{cases}$$

where $\theta_0 = .64\ldots \cdot \pi$ (almost $2\pi/3$) is the solution to $1 - 2\cos\theta = \mathcal{R}(\theta)$ and $\theta_1 = .53\ldots \cdot \pi$ (just above $\pi/2$) is the solution to $\mathcal{R}(\theta) = \mathcal{S}(\theta)$. These results extend to entire functions.

That means if $p = \sum_{i=0}^{N} c_i z^i$ ($N \in \{6, 7, 8, \dots\} \cup \{\infty\}$) is an entire function with only nonzero coefficients and satisfying $c_i^2/c_{i+1}c_{i-1} \geq \beta_0(\theta)$ for some θ but all $1 \leq i < N$, then for all zeros z of p, $|\arg z| > \theta$, and $\beta_0(\theta)$ is the smallest number with this property. For example, if $\theta = 3\pi/4$, $2\pi/3$, $\pi/2$, the respective values of $\beta_0(\theta)$ are $1 + \sqrt{2}$, 2, and the real root of $X^3 - X^2 - 1 = 0$ ($((116 + 12 \cdot 93^{1/2})^{1/3} + 4(116 + 12 \cdot 93^{1/2})^{-1/3} + 2)/6$; approximately $1.46557\ldots$).

Now we have the relatively easy necessary conditions.

Recall that $\mathcal{S}(\theta)$ is the largest positive root of $X^3 - (1+a)X^2 - 1 + a + a^2$ where $a = 2\cos\psi$, when it exists, and 1 otherwise.

LEMMA 1.5 With $N \geq 5$ and $\pi/2 \leq \theta < \pi$ and $\psi = \pi - \theta$, we have that

$$\beta_0(\theta) \geq \max\left\{4\cos^2\theta, 1 + 2\cos\psi, \mathcal{R}(\theta), \mathcal{S}(\theta)\right\}.$$

Proof. We first obtain polynomials of degrees 2 through 5 yielding the lower bounds, and then show how they can be enlarged to sequences of polynomials of degree N to yield the lower bounds in all cases. For convenience, let $a = 2\cos\psi$. Set $g := z^2 + az + 1$; its roots are $e^{\pm i\theta}$, and its lone β_1 is $4\cos^2\psi$. Now set $h = g \cdot (1+z) = z^3 + (1+a)z^2 + (1+a)z + 1$, whose β values are $1+a$, and obviously with roots $\left\{e^{\pm i\theta}, -1\right\}$.

Next, set $j = g \cdot (1 + bz + z^2)$ where b is to be determined. This expands as $z^4 + (a+b)z^3 + (2 + ab)z^2 + (a+b)z + 1$, whose β-values are $\left\{(a+b)^2/(2+ab), (2+ab)^2/(a+b)^2\right\}$. Let b be a positive root of the equation $(a+b)^2/(2+ab), = (2+ab)^2/(a+b)^2$, i.e., $(a+b)^4 = (2+ab)^3$ (if none exist, then this will correspond to the value 1 for $\mathcal{R}(\theta)$, so that the sole β-value of j is $(a+b)^{2/3} := \beta$. This yields $b = \beta^{3/2} - a$, and substituting this into the equation, we obtain $\beta^6 = (2 + a(\beta^{3/2} - a)^3$, or in other words, $\beta^2 - a\beta^{3/2} + a^2 - 2 = 0$. By going in reverse, we reconstruct b from the positive real root of this quartic. Hence the β value of j is $\mathcal{R}(\theta)$.

Finally, set $k = g \cdot (1 + bz + bz^2 + z^3)$, again with b to be determined. Here $k = z^5 + (a+b)z^4 + (1+b+ab)z^3 + (1+b+ab)z^2 + (a+b)z + 1$, with at most two distinct β values, $(a+b)^2/(1+b+ab)$ and $(1+b+ab)/(a+b)$; equating them as in the previous case, we obtain the equation $(a+b)^3 = (1+ab+b)^2$; if this has a positive real root b, the β value of the corresponding choice of k is $\beta := (a+b)^{1/2}$. Then with the substitution $b = \beta^2 - a$ yields the equation (for β) $\beta^6 = (1 + (\beta^2 - a)(1+a))^2$, and since both sides are positive, $\beta^3 = 1 + \beta^2(1+a) - a - a^2$, or in other words, $\beta^3 - (1+a)\beta^2 - 1 + a + a^2 = 0$. Now start with this and define b to construct k.

In each of the four cases we have found polynomials (of degrees two through five) whose β values are as indicated, and have $e^{i\theta}$ as a root. If $N \geq 5$ (or $N > 5$ and we are dealing $\mathcal{S}(\theta)$), let l be one of $\{2,3,4,5\}$ and define for each integer n, $f_n = 1 + z^1/n^2 + z^2/n^6 + \cdots + z^{N-l}/n^{N(N-l)}$ (so that the β values are all $1/n^2$, except if $N - l = 1$). If F is any polynomial whose minimal β value larger than 1, it is easy to see that the minimal beta values of the elements of the sequence $F \cdot f_n$ converge to that of F. For $F \in \{g,h,j,k\}$, $e^{i\theta}$ is a root of F hence of $F \cdot f_n$ for all n, and since the degree of $F \cdot f_n$ is N, it follows that $\beta_0(\theta)$ is at least the β-value of each $F \cdot f_n$, hence is at least the limit of their β values. •

We will frequently work with $\psi = \pi - \theta$. The proof that $\beta_0(\theta)$ is bounded above by the right side involves overlapping intervals on which we work with only two of the terms to be maximized at a time (e.g., on $(3\pi/4, \pi)$, we show $\beta_0(\theta) \leq \max\left\{4\cos^2\theta, 1 - 2\cos\theta\right\}$). The reverse inequalities (which shows that the right side is always sharp) are obtained from tricky multiplications, which are easy to implement, but were difficult to find.

We note the following elementary inequalities. Suppose $f = \sum c_j z^j$ belongs to U_β, and k is a mode for the sequence (c_j). Then with $l \geq 0$ and j positive and large enough,

$$\begin{aligned}
\frac{c_{k+l+j}}{c_{k+l+j-1}} &= \frac{c_{k+l+j-1}}{c_{k+l+j-2}} \cdot \frac{1}{\beta_{k+l+j-1}} \leq \frac{c_{k+l+j-1}}{c_{k+l+j-2}} \cdot \frac{1}{\beta} \leq \cdots \\
&\leq \frac{c_{k+l}}{c_{k+l-1}} \cdot \frac{1}{\beta^j} \qquad \text{and} \\
\frac{c_{k+l+j}}{c_{k+l+j-1}} &\leq \frac{c_{k+1}}{c_k} \cdot \frac{1}{\beta^{j+l-1}}
\end{aligned}$$

and

$$\begin{aligned}\frac{c_{k+l+j}}{c_{k+l}} &= \frac{c_{k+l+j}}{c_{k+l+j-1}}\frac{c_{k+l+j-1}}{c_{k+l+j-2}}\cdot\ldots\cdot\frac{c_{k+l+1}}{c_{k+l}}\\ &\leq \left(\frac{c_{k+l+1}}{c_{k+l}}\right)^j\cdot\frac{1}{\beta^{(j^2+j)/2}}\end{aligned}$$

Throughout, $w = e^{i\theta}$ and $\psi = \pi - \theta$.

THEOREM 1.6 With fixed $N \geq 6$, we have the following.

$$\beta_0(\theta) = \begin{cases} 4\cos^2\theta & \text{if } 4\pi/5 \leq \theta < \pi \\ 1 - 2\cos\theta & \text{if } \theta_0 \leq \theta \leq 4\pi/5 \\ \mathcal{R}(\theta) & \text{if } \theta_1 \leq \theta \leq \theta_0 \\ \mathcal{S}(\theta) & \text{if } \pi/2 \leq \theta \leq \theta_1 \end{cases}$$

where $\theta_0 = .64\ldots\cdot\pi$ is the solution to $1 - 2\cos\theta = \mathcal{R}(\theta)$ and $\theta_1 = .53\ldots\cdot\pi$ is the solution to $\mathcal{R}(\theta) = \mathcal{S}(\theta)$.

In particular, if $f = \sum_{j=0}^N c_j z^j$ with $N \in \{6,7,8,\ldots\}\cup\{\infty\}$ is entire with all coefficients nonzero and nonnegative, and $\inf_{1\leq j<N} c_j^2/c_{j+1}c_{j-1} \geq \max\left\{4\cos^2\theta, 1-2\cos\theta, \mathcal{R}(\theta), \mathcal{S}(\theta)\right\}$ for some θ in $[\pi/2,\pi)$, then all zeros of f lie in the open sector $|\arg z| > \theta$.

Proof. We fix $6 \leq N < \infty$. By 1.4, we need only obtain conditions excluding zeros at various $e^{i\theta}$—that is, we show that if β is at least as large as the right side (usually treated two at a time), and f is in U_β, then $f(e^{i\theta}) \neq 0$. This yields that $\beta_0(\theta)$ is bounded above by the right side, but we already have the reverse inequality in 1.5. We deal with three overlapping intervals.
(a) $3\pi/4 \leq \theta < \pi$. Here we show that $\beta_0(\theta) \geq \max\left\{4\cos^2\phi, 1-2\cos\phi\right\}$.
Suppose there exists f in U_β (β as yet unspecified—we wish to derive conditions that guarantee a contradiction, which will typically bound β) such that $f(e^{i\theta}) = 0$. By replacing f by its opposite if necessary, we may assume that the coefficient to the right of the mode is less than or equal to the coefficient to the left of the mode. Let k be the mode, so that $c_{k+1} \leq c_{k-1}$, and consider $-\mathrm{Im}\,(e^{-(k-2)i\theta}f(e^{i\theta}))$. This expands (replacing θ by $\pi-\psi$, which makes the manipulations clearer; here $0 < \psi \leq \pi/4$) as

$$\begin{aligned}&[\cdots - c_{k-4}\sin 2\psi + c_{k-3}\sin\psi + 0\cdot c_{k-2}]\\ &\qquad + [-c_{k-1}\sin\psi + c_k\sin 2\psi - c_{k+1}\sin 3\psi]\\ &\qquad + [c_{k+2}\sin 4\psi - c_{k+3}\sin 5\psi + \ldots];\end{aligned}$$

we interpret $c_{\text{negative}} = 0$. We will analyze this in three parts, the middle three terms, $-c_{k-1}\sin\psi + c_k\sin 2\psi - c_{k+1}\sin 3\psi$, the right tail, $c_{k+2}\sin 4\psi - c_{k+3}\sin 5\psi + \ldots$, and the left tail, $\cdots - c_{k-4}\sin 2\psi + c_{k-3}\sin\psi$. We will show that if $\beta \geq \max\left\{4\cos^2\psi, 1+2\cos\psi\right\}$, then all three are nonnegative, and at least one of the tails is positive (of course, depending on k and N, the left or right tail might not even exist). The computation of the middle term brings out the connection with the necessary conditions, while the tails just require estimates on the rate of decay (which is very rapid).

Set $r = c_{k-1}/c_k$ and $s = c_{k+1}/c_k$, so that $0 \leq s \leq r \leq 1$ and $rs \leq 1/\beta$. To show the middle term is nonnegative, it suffices to show that

$$\max\{r\sin\psi + s\sin 3\psi \mid 0 \leq r \leq s \leq 1,\ rs \leq 1/\beta\} \leq \sin 2\psi.$$

It is immediate that the only two locations for the maximum value of the left side occur at $(r,s) = (\beta^{-1/2}, \beta^{-1/2})$ and $(1, 1/\beta)$.

The value at the former leads to $\beta^{-1/2}(\sin\psi + \sin 3\psi) \leq \sin 2\psi$. Since $\sin\psi + \sin 3\psi = 2\sin 2\psi \cos\psi$, we obtain $\beta^{-1} \leq 1/2\cos\psi$, or $\beta \geq 4\cos^2\psi$ (that was easy).

The latter leads to $\beta^{-1} \leq (\sin 2\psi - \sin\psi)/\sin 3\psi$. From $\sin 2\psi = 2\sin\psi\cos\psi$ and $\sin 3\psi = 3\sin\psi - 4\sin^3\psi = \sin\psi(4\cos^2\psi - 1)$, we see that the inequality is equivalent to $\beta^{-1} \leq (2\cos\psi - 1)/(4\cos^2\psi - 1) = 1/(2\cos\psi + 1)$.

Thus if $\beta \geq \max\left\{4\cos^2\psi, 1 + 2\cos\psi\right\}$ (that is, both inequalities occur), then the middle term is nonnegative.

Now we look at the right tail. We begin by supposing that $\pi/(K+1) \leq \psi < \pi/K$ for some positive integer $K \geq 5$ (we have to consider the case that $K = 4$ separately). Then $\sin 4\psi, \sin 5\psi, \ldots, \sin(K+1)\psi$ are all nonnegative, and thus the sequence that appears in the expansion of the tail, that is,

$$c_{k+2}\sin 4\psi, -c_{k+3}\sin 5\psi, c_{k+4}\sin 6\psi, \ldots, (-1)^{K+1}c_{k+K-1}\sin(K+1)\psi$$

is alternating. We will show that the absolute values are decreasing, and so get a good lower bound for this portion of the tail. The remainder of the tail is so small that it is easy to deal with.

The ratio of absolute values of consecutive terms of $(-1)^j c_{k+j}\sin(j+2)\psi$ ($j = 2, 3, \ldots, K+1$) is $(c_{k+j+1}/c_{k+j})(\sin(j+3)\psi/\sin(j+2)\psi)$. The left factor is bounded above by $s/\beta^j \leq \beta^{-j-1/2}$, and the right term is bounded above by 2 (note that for $j > (K-2)/2$, each $\sin(j+3)\psi/\sin(j+2)\psi$ is less than one. On the interval for ψ, β is at least $\max\left\{4\cos^2\pi/K, 1 + 2\cos\pi/K\right\} > 2.5$. Since $\beta^{-j-1/2} < 1/2$, the sequence is descending.

Hence the sum of the alternating sequence $\sum_{j=2}^{K-1}(-1)^j c_{k+j}\sin(j+2)\psi$ is bounded above by $c_{k+2}\sin 4\psi - c_{k+3}\sin 5\psi$. This can be rewritten as $c_{k+2}\sin 4\psi(1 - (c_{k+3}/c_{k+2})(\sin 5\psi/\sin 4\psi)) \leq c_{k+2}\sin 4\psi(1 - (\sin 5\psi/\sin 4\psi)\beta^{-5/2}$. The remainder of the tail can be bounded in absolute value by

$$\begin{aligned}
\sum_{j=0}^{\infty} c_{k+K+j}|\sin(K+j+2)\psi| &\leq \sum_{j=0}^{\infty} c_{k+K+j}\\
&\leq c_{k+2}\sum_{j=0}^{\infty}\frac{c_{k+K+j}}{c_{k+2}}\\
&\leq c_{k+2}\sum_{j=0}^{\infty}\beta^{-((K+j)^2+(K+j)-6)/2}
\end{aligned}$$

The series is $\beta^{(-K^2-K+6)/2} + \beta^{(-K^2-3K+5)/2} + \ldots$ which is bounded above by $(5/4)\beta^{(-K^2-K+6)/2}$. So all we need is that $\sin 4\psi(1 - (\sin 5\psi/\sin 4\psi)\beta^{-5/2}) > (5/4)\beta^{(-K^2-K+6)/2}$ for $K \geq 5$ and ψ in the interval. Sufficient is that $\sin 4\psi > 2.4^{(-K^2-K+6)/2}$. Since $\sin 4\psi > \psi > \pi/K$ on this interval, the result follows easily.

Now suppose (still dealing with the right tail) that $\pi/5 \leq \psi \leq \pi/4$. In this case, $\sin 5\psi$ is negative, and the minimum of $c_{k+2}\sin 4\psi - c_{k+3}\sin 5\psi$ occurs at the right endpoint (easy to check, since $c_{k+3}/c_{k+2} \leq s/\beta^2 < 1/2$), which is $\sqrt{3}c_{k+3}/2$. The remainder of the tail is bounded by $\sum_{j\geq 4} c_{k+j}$, which is dealt with as above.

The treatment of the left tail is similar, but a little simpler. Again assume that $\pi/(K+1) \leq \psi \leq \pi/K$. The sequence $\sin\psi, -\sin 2\psi, \ldots, (-1)^{K-1}\sin K\psi$ is alternating and $c_{k-j-2}\sin j\psi$ (for

$j = 1, \ldots, K$) is descending, so

$$\sum_{j=1}^{K} (-1)^{j-1} c_{k-j-2} \sin j\psi \geq c_{k-3} \sin\psi - c_{k-4} \sin 2\psi$$
$$\geq c_{k-3}(\sin\psi - \sin 2\psi/\beta^3) = c_{k-3} \sin\psi(1 - 2\beta^{-3}\cos\psi)$$
$$\geq c_{k-3} \sin\frac{\pi}{K+1}(1 - 2\beta^{-3}).$$

The remaining terms in the left tail are bounded above by $\sum_{j=0} c_{k-K-2-j}$, which as before, is bounded above by $c_{k-3}\left(\beta^{(-K^2+K)/2} + \beta^{(-K^2-K)/2} + \ldots\right)$. All that remains is to verify that $(1-2\beta^{-3})\sin\pi/(K+1) > 1.1\beta^{(-K^2+K)/2}$ for $\beta > 2.4$ and $K \geq 4$, which is easy.
(b) $3\pi/5 \leq \theta \leq 3\pi/4$, i.e., $\pi/4 \leq \psi \geq 2\pi/5$. Here we show that $\beta_0(\theta) \leq \max\{1 + 2\cos\psi, \mathcal{R}(\theta)\}$ Recall that $\mathcal{R}(\theta)$ is the largest positive real zero of (what is effectively) a quartic, $X^2 + X^{3/2} 2\cos\theta + 2\cos 2\theta = 0$; if none exist, define $\mathcal{R}(\theta) = 1$.

On this interval, $\sin\psi$ and $\sin 2\psi$ exceed zero, and $\sin 4\psi$ is nonpositive; $\sin 5\psi < 0$ if $\psi < 2\pi/5$. Let k be the mode of $\{c_k\}$ and assume that $c_{k-1} \leq c_{k+1}$. Consider the middle terms, $-c_{k-1}\sin\psi + c_k \sin 2\psi - c_{k+1}\sin 3\psi + c_{k+2}\sin 4\psi$ of $\operatorname{Im} w^{-k+2} f(w)$; set $r = c_{k-1}/c_k$ and $s = c_{k+1}/c_k$, so that $0 \leq r \leq s \leq 1$ and $rs \leq 1/\beta$. We derive conditions (on β) to guarantee that $-r\sin\psi + \sin 2\psi - s\sin 3\psi + 4s^2/\beta \sin 4\psi$ is nonnegative (which is sufficient for nonnegativity of the middle term since $c_{k+2} \leq s^2 c_k/\beta$). This amounts to $r + s(4\cos^2\psi - 1) + s^2|8\cos^3\psi - 4\cos\psi|/\beta \leq 2\cos\psi$. Now the maximum value of the left side occurs at either $(r,s) = \beta^{-1/2}(1,1)$ or at $(1/\beta, 1)$. The former yields a maximum value of $\beta^{-1/2}(4\cos^2\psi) - 4\cos\psi(2\cos^2\psi - 1)\beta^{-2}$, which is less than or equal to the right side if and only if $\beta^2 - \beta^{3/2} 2\cos\psi + 2\cos 2\psi \geq 0$, that is, $\beta \geq \mathcal{R}(\theta)$ (in converting between θ and ψ, the middle term is multiplied by -1, but not the constant term).

At $(r,s) = (1/\beta, 1)$, we require $\beta^{-1}(1 - 4\cos\psi(2\cos^2\psi - 1)) \leq 1 + 2\cos\psi - 4\cos^2\psi$, that is $\beta^{-1} \leq (1 + 2\cos\psi - 4\cos^2\psi)/(1 - 4\cos\psi(2\cos^2\psi - 1)) = 1/(1 + 2\cos\psi)$. Hence $\beta \geq \max\{1 + 2\cos\psi, \mathcal{R}(\theta)\}$ is sufficient to guarantee that this middle cluster of terms is nonnegative. Now we deal with the tails.

Now we deal with the right tail. The tail begins $-c_{k+3}\sin 5\psi + c_{k+4}\sin 6\psi - 7c_{k+5}\sin 7\psi + \ldots$. The first subcase is $\pi/4 \leq \psi \leq \pi/3$. On this interval, the leading term, $-c_{k+3}\sin 5\psi$ is at least $c_{k+3}/\sqrt{2}$. The rest of the tail is bounded in absolute value rather crudely by $\sum_{j\geq 1} c_{k+3+j}$. Obviously

$$\sum_{j\geq 1} c_{k+3+j} = c_{k+3} \sum_{j\geq 1} \frac{c_{k+3+j}}{c_{k+3}} \leq c_{k+3} \sum_{j\geq 1} \frac{s^j}{\beta^{(j^2+5j)/2}}$$
$$= c_{k+3}\left(\frac{1}{\beta^3} + \frac{1}{\beta^7} + \frac{1}{\beta^{12}} + \ldots\right).$$

With $\beta \geq 2^{1/2}$, we deduce $-c_{k+3}\sin 5\psi > \sum c_{k+3+j}$, hence the right tail is positive.

The next subcase assumes $\pi/3 \leq \psi \leq 2\pi/5$. Then the leading two terms are nonnegative, and

$$-c_{k+3}\sin 5\psi + c_{k+4}\sin 6\psi \geq c_{k+4}(\sin 6\psi - \sin 5\psi) > .9c_{k+4}$$

Now the rest of the tail is bounded in absolute value by $\sum_{j\geq 1} c_{k+4+j}$, and the same technique as in the previous subcase yields that the right tail is positive.

For the left tail, $c_{k-3}\sin\psi - c_{k-4}\sin 2\psi + c_{k-5}\sin 3\psi - \ldots$, the leading term is at least $c_{k-3}/\sqrt{2}$, and the same argument as in the first subcase of the right tail (but without requiring further restrictions on ψ) will work.

(c) $13\pi/36 \leq \psi \leq \pi/2$.
Here we show that $\beta_0(\theta) \leq \max\{\mathcal{R}(\theta), \mathcal{S}(\theta)\}$.
We note the following identities

$$\begin{aligned}\sin\frac{3\psi}{2} &= \sin\frac{\psi}{2}\,(1+2\cos\psi)\\ \sin\frac{5\psi}{2} &= \sin\frac{\psi}{2}\left(4\cos^2\psi+2\cos\psi-1\right)\end{aligned}$$

We may assume the mode appears at k and $c_{k-1} \leq c_{k+1}$. Consider $-\operatorname{Im} w^{-k+3} f(w)$, which expands as

$$\begin{aligned}&[\cdots - c_{k-4}\sin 7\psi + c_{k-3}\sin 6\psi]\\ &\qquad + [-c_{k-2}\sin 5\psi + c_{k-1}\sin 4\psi - c_k \sin 3\psi + c_{k+1}\sin 2\psi - c_{k+2}\sin\psi]\\ &\qquad + [c_{k+4}\sin\psi - c_{k+5}\sin 2\psi + \dots]\end{aligned}$$

(Note that $\sin 4\psi$ and $\sin 3\psi$ are both nonpositive and $\sin 6\psi$ is positive on $\pi/3 < \psi \leq \pi/2$.) As usual, begin with the middle thing; with $r = c_{k-1}/c_k$ and $s = c_{k+1}/c_k$, it is sufficient to determine conditions on β to guarantee that

$$-\frac{r^2}{\beta}\sin 5\psi + \frac{r}{\beta}\sin 4\psi - \sin 3\psi + \frac{s}{\beta}\sin 2\psi - \frac{s^2}{\beta}\sin\psi \geq 0,$$

subject to the constraints $0 \leq r \leq s \leq 1$ and $rs \leq 1/\beta$. It is again easy to check that the minimum value occurs at the the vertices of the domain, $(r,s) = \beta^{-1/2}(1,1)$ and $(\beta^{-1},1)$. Evaluating at the first yields

$$\frac{-1}{\beta^2}(\sin 5\psi + \sin\psi) + \frac{1}{\sqrt{\beta}}(\sin 4\psi + \sin\psi) - \sin 3\psi.$$

Noting that $-\sin 3\psi > 0$ (if $\pi/3 < \psi \leq \pi/2$), we may divide by $(-\sin 3\psi)\beta^2$, and expanding the sums of sines, we obtain the equivalent inequality, $\beta^2 - 2\beta^{3/2}\cos\psi + 2\cos 2\psi \geq 0$, for which $\beta \geq \mathcal{R}(\theta)$ is sufficient.

At the point $(r,s) = (1/\beta, 1)$, we obtain $\sin 2\psi - \sin 3\psi + \beta^{-1}\cdot(\sin 4\psi - \sin\psi) - \beta^{-3}\cdot \sin 5\psi$. Converting the differences of sines and noting that $-\cos 5\psi/2 > 0$, we can divide this by $-2\cos(5\psi/2)\sin\psi/2\beta^{-3}$, and obtain the equivalent,

$$\beta^3 - \frac{\sin 3\psi/2}{\sin\psi/2}\beta^2 + \frac{\sin 5\psi/2}{\sin\psi/2} \geq 0.$$

For this, $\beta \geq \mathcal{S}(\theta)$ is sufficient, by the identities above.

Hence if $\beta \geq \max\{\mathcal{R}(\theta), \mathcal{S}(\theta)\}$, the middle part is nonnegative; this was under the assumption that $\pi/3 < \psi \leq \pi/2$. Now we deal with the tails.

For the left tail, we first consider $13\pi/36 \leq \psi \leq 3\pi/7$. Then $13\pi/6 \leq 6\psi \leq 18/7\pi$, so $\sin 6\psi \geq 1/2$. The rest of the tail is bounded in absolute value by $\sum_{j\geq 0} c_{k-4-j}$, which as usual is bounded above by $c_{k-3}(\beta^{-3} + \beta^{-6} + \beta^{-10} + \dots$. Since $\beta > \sqrt{2}$, the factor is less than $1/2$, and so the left tail is positive.

Next, assume $3\pi/7 \leq \psi \leq \pi/2$. Then $3\pi \leq 7\psi \leq 7\pi/2$, so $-\sin 7\psi \geq 0$. Hence

$$c_{k-3}\sin 6\psi - c_{k-4}\sin 7\psi \geq c_{k-4}(\sin 6\psi - \sin 7\psi) = 2c_{k-4}\sin\frac{\psi}{2}\left|\cos\frac{13\psi}{2}\right| > .85c_{k-4}.$$

The rest of the tail is bounded above in absolute value by $\sum_{j\geq 0} c_{k-5-j} \leq c_{k-4}(\beta^{-4}+\beta^{-7}+\beta^{-11}+\dots)$, which is a lot less than $.85c_{k-3}$ (when $\beta > \sqrt{2}$).

For the right tail, we note that $\sin\psi > .95$ on the entire interval $[3\pi/7, \pi/2]$ ($\sin 3\pi/7 = .9749\dots$), and the rest of the tail is bounded above by $c_{k+4}(\beta^{-4}+\beta^{-7}+\dots) < .5c_{k+4}$.

Hence we have that (with fixed finite $N \geq 6$), $\beta_0(\theta)$ is given by the right side. When N is finite, the final statement is a restatement of the definition of $\beta_0(\theta)$. When N is infinite, we note that the finite truncations converge uniformly on compact sets to f, and each of these truncations satisfy the same conditions, hence their zeros lie within W_θ, and thus so do all the zeros of f. •

Owing to the awkward definitions of $\mathcal{R}(\theta)$ and $\mathcal{S}(\theta)$, it is worthwhile discussing the function inverse to $\beta_0 : [\pi/2, \pi) \to [1.46557\dots, 4]$ (β_0 is of course defined on all of $(0,\pi)$, but we only have an exact formula available on $([\pi/2,\pi))$.

Let P_N^{++} denote the set of polynomials of degree N all of whose coefficients are strictly positive; obviously $U_\beta \subset P_N^{++}$ for all $\beta > 1$. Taking, as usual the branch of $\arg z$ given by $-\pi < \arg z \leq \pi$, define

$$T : P_N^{++} \to (0,\pi] \qquad \text{by} \qquad T(f) = \inf\{|\arg z| \mid z \in Z(f)\}.$$

Since polynomials in P_N^{++} have no positive real numbers as zeros, T is well-defined. A simple Rouch-convergence argument shows that T is continuous.

Define $\Theta : (1,4] \to [0,\pi]$ via

$$\Theta(\beta) = \sup\{\theta \in (0,\pi] \mid Z(f) \subset W_\theta \text{ for all } f \in U_\beta\}.$$

It is easy to check that $\Theta = \beta_0^{-1}$, and moreover, $\Theta(\beta) = \inf\{T(f) \mid f \in U_\beta\}$. On the interval $[1.46557\dots, 4]$ (the left endpoint is the real root of $X^3 - X^2 - 1 = 0$), we have (where $\gamma = (1+\sqrt{5})/2$ is the golden ratio)

$$2\cos\Theta(\beta) = \begin{cases} -\sqrt{\beta} & \text{if } \gamma^2 \leq \beta \leq 4 \\ 1-\beta & \text{if } 1.57762\dots \leq \beta \leq \gamma^2 \\ -\beta^{3/2} + \sqrt{\beta^3 - 4\beta^2 + 8} & \text{if } 1.52334\dots \leq \beta \leq 1.57762\dots \\ 1-\beta^2 + \sqrt{(1+\beta^2)^2 + 4(1-\beta^3)} & \text{if } 1.46557\dots \leq \beta \leq 1.52334\dots \end{cases}$$

If $\theta < \pi/2$. When $\theta < \pi/2$, especially as $\theta \to 0$, new phenomena occur. The first is that the dependence on N (which was barely noticeable up to this point) becomes more marked. It is worthwhile giving an equivalent form of [H, Corollary 1.3].

PROPOSITION 1.7 *Suppose that q is a monic polynomial of degree $n-1$ or less, and has real nonnegative coefficients which form a unimodal sequence. If q has a zero at the single point $\exp(2\pi i/n)$, then $g = 1 + z + z^2 + \dots + z^{n-1}$.*

In this result, q is assumed to vanish only at the single primitive root of unity, $e^{2\pi i/n}$, from which (together with unimodality) we deduce that it vanishes at *all* nth roots of unity. From this, in order to obtain meaningful results about $\beta_0(\theta)$ for $\theta = 2\pi/n$ (and for values close to this), we must assume that the corresponding N is at least n, and likely at least $3n/2$. For that reason, we redefine $\beta_0(\theta)$ in what follows to be the $\liminf_{N\to\infty}$ of the previously-defined $\beta_0(\theta)$. It is likely that for fixed θ, the sequence is ultimately stationary.

Lower bounds for the values of β_0 (for $\theta < \pi/2$) seem intractible at the moment—they likely involve multiplication by polynomials of increasing degree (see the proof of 1.4). On the other hand, as $\theta \to 0$, we can obtain asymptotic estimates for $\beta_0(\theta) - 1$; specifically, $\beta_0(\theta) - 1 \leq 16\theta^2 \ln 2/\pi^2$

(the constant, $16\ln 2/\pi^2$, is rather flabby, and doubtless can be improved), and it is relatively easy to see that $\beta_0(\theta) - 1 \geq \theta^2/4\pi^2$. Hence $\beta_0(\theta)-1$ is bounded above and below by a multiple of θ^2.

As β_0 is monotone, modulo a bit of fiddling with the scalar multiple, to prove the first statement, $\beta_0(\theta)-1 \leq K\theta^2$, it suffices to do it with $\theta = \pi/n$, where n is a positive integer. We illustrate the case that 4 divides n; the other cases are very similar.

$\theta = \pi/n$ and 4 divides n. Assume $\beta^{n^2/2} \geq 2$ (this will be subsumed by a stronger condition), mode at k; form $\operatorname{Im} w^{-k+n/2} f(w)$; then the middle clump in the expansion consists of about $3n$ terms,

$$\sum_{j=-n/2}^{n/2} c_{k+j} \sin(n/2+j)\pi/n - \sum_{j=-3n/2}^{-n/2-1} c_{k+j} \sin(3n/2+j)\pi/n - \sum_{j=n/2+1}^{3n/2-1} c_{k+j} \sin(3n/2-j)\pi/n$$

where we have indexed the arguments of the sine in order to ensure that the sine terms are nonnegative. Now we show nonnegativity of each of the following terms,

$$\begin{gathered}
c_{k-n/2+1} \sin \pi/n - (c_{k-3n/2+1} \sin \pi/n + c_{k-3n/2+2} \sin 2\pi/n) \\
c_{k-n/2+2} \sin 2\pi/n - (c_{k-3n/2+3} \sin 3\pi/n + c_{k-3n/2+4} \sin 4\pi/n) \\
\ldots\ldots \\
c_{k-n/2+l} \sin l\pi/n - (c_{k-3n/2+2l-1} \sin(2l-1)\pi/n + c_{k-3n/2+2l} \sin 2l\pi/n) \\
\ldots\ldots \\
c_{k-n/4} \sin \pi/4 - (c_{k-n-1} \sin(n/2-1)\pi/n + c_{k-n} \sin \pi/2)
\end{gathered}$$

We note that

$$\begin{aligned}
\frac{c_{k-3n/2+2l-1}}{c_{k-n/2+l}} &\leq \frac{1}{\beta^{(n-2l+1)(n-2l+2)/2}} \\
\frac{c_{k-3n/2+2l}}{c_{k-n/2+l}} &\leq \frac{1}{\beta^{(n-2l+2)(n-2l+3)/2}}
\end{aligned}$$

Hence if $e(l) = n-2l)^2 + 3(n-2l)+2)/2$, we need only show that for $l = 1, \ldots, n/4$,

$$\beta^{e(l)} \geq \frac{\sin(2l-1)\pi/n + \sin 2l\pi/n}{\sin l\pi/n}.$$

The right side is always less than 4, so sufficient is $\beta^{e(l)} \geq 4$, as occurs if $\beta^{n^2/8+3n/4+1} \geq 4$. Hence $\ln\beta \geq 2\ln 2/(n^2/8+3n/4)$, or merely $\ln\beta \geq (16\ln 2)/n^2$ is sufficient for this collection of inequalities.

The next batch of inequalities is treated similarly, but there is a slight difference. We consider the following,

$$c_{k-n/4+l} \sin\left(\frac{\pi}{4} + \frac{\pi l}{n}\right) - c_{k-n+2l-1} \sin\left(\frac{\pi}{2} + \frac{\pi(2l-1)}{n}\right) - c_{k-n+2l} \sin\left(\frac{\pi}{2} + \frac{2\pi l}{n}\right), \quad l = 1, 2, \ldots, \frac{n}{4} - 1.$$

The claim, as above, is that when β sufficiently large, these are all nonnegative. We have that $c_{k-n+2l-1}/c_{k-n/4+l} \leq \beta^{-e(l)}$, where $e(l) = 15n^2/32 - 7nl/4 + n + 3(l^2-l)$. The smallest value of $e(l)$ occurs (over the range $1 \leq l \leq n/4-1$) when $l = n/4-1$, and we see that $e(l) > n^2/8$. We also have that the ratio of the sines is bounded above by $\sqrt{2}$ (some of the flabbiness creeps in here), so sufficient for all the inequalities to hold is that $\beta^{n^2/8} > 2\sqrt{2}$, that is, $\ln\beta > 12\ln 2/n^2$ and in fact these are all strict. We have c_k left over (which does not happen in the other cases, that is, when n is not divisible by four).

Thus sufficient for the middle block of $3n$ or so terms to be nonnegative, it is sufficient that $\ln \beta \leq 16 \ln 2/n^2$. The remainder of the expansion is block alternating, and monotone in each of the n positions, hence the outcome is nonnegative.

In particular, $\beta_0(\pi/n) < \exp(16 \ln 2/n^2) \sim 1 + 16 \ln 2/n^2$.

To give a rough upper bound for $\beta_0(\pi/n)$, we simply note that $C_{2n} := (1 - z^{2n})/(1 - z) = \sum_{0 \leq j \leq 2n-1} z^j$ has $\exp(2\pi i/2n)$ as a root and $C_n C_{2n}$ has sequence of coefficients $(1, 2, 3, \ldots, n, n, n-1, \ldots, 1)$. The minimum $c_j^2/c_{j-1}c_{j+1}$ occurs when $j = n-2$, i.e., $(n-1)^2/n(n-2) = 1+1/n(n-2)$. Hence $\beta_0(2\pi/n) - 1 > 1/n^2$, so $\beta_0(\pi/n) - 1 > 1/4n^2$.

Both the upper and lower estimates were obtained rather sloppily, and it is very unlikely that either one is even close to being sharp.

Section 2. Constant quadratic ratios

Define for N a positive integer, the polynomial of degree N, $f_{b,N} = \sum_{j=0}^{N} x^j b^{-j(j+1)}$; if N is infinite, the resulting series is entire. This satisfies the property that $c_j^2/c_{j+1}c_{j-1} = b^2$ for all $1 \leq j < N$. We will determine to within 10^{-24}, the minimum of the b such that $f_{b,N}$ has only real zeros, for sufficiently large N, and also for infinite N.

We define the *opposite* of a polynomial of degree N to be the polynomial with coefficients written in reverse order; explcitly, $f^{\mathrm{op}}(x) = x^N f(x^{-1})$. A polynomial is *symmetric* (or *self-reciprocal*) if $f^{\mathrm{op}} = f$. If f is symmetric of degree N, then $f(x) = x^N f(x^{-1})$, and in particular, the set of zeros of f are closed under the operation $w \mapsto 1/w$. Every real polynomial $f = \sum_{j=0}^{N} c_j z^j$ with all $c_j^2/c_{j+1}c_{j-1}$ equal, say to $\beta > 0$ for $1 \leq j \leq N-1$ can be reparameterized so as to be symmetric. For N odd (that is, an even number of coefficients), up to scalar multiple, the distribution of coefficients of a symmetric polynomial satisfying this condition $(\ldots \beta^{-3}\ \beta^{-1}\ 1\ 1\ \beta^{-1}\ \beta^{-3}\ \ldots)$ (the exponents are triangular numbers), while for N even, the distribution is $(\ldots \beta^{-9/2}\ \beta^{-2}\ \beta^{-1/2}\ 1\ \beta^{-1/2}\ \beta^{2}\ \beta^{-9/2}\ \ldots)$ (the exponents are half-squares).

Form $\mathcal{F} \equiv \mathcal{F}_{b,N}(x) = f_{b,N}(xb^{N+1})$ (for which $c_j^2/c_{j+1}c_{j-1} = b^2$ for all relevant j) and write $N = 2r-1$ (N odd) or $N = 2r$ (if N is even). We see that the ratio of the coefficient of x^r in $\mathcal{F}$ to that of x^{r-1} is 1 if N is odd, and is b if N is even; since $\mathcal{F}$ also has the property that all the ratios $c_j^2/c_{j+1}c_{j-1}$ are equal to b^2, this enough to guarantee that $\mathcal{F}$ is symmetric. In particular, for all real $x > 0$, $\operatorname{sign}\left(\mathcal{F}_{b,N}(-x^{-1})\right) = (-1)^N \operatorname{sign}\left(\mathcal{F}_{b,N}(-x)\right)$.

For l a positive integer less than or equal to N, consider $f_{b,l}$; its list of consecutive coefficients is just an initial segment of that of $f_{b,N}$.

PROPOSITION 2.1 Suppose that $b > \sqrt{3}$, that l is a positive integer, and that $N \geq 2l$ is an integer.

(a) If l is even and there exists a positive real number x_1 such that $1 < x_1 < b^4$ and $f_{b,l}(-x) \leq 0$, then for all $b' \geq b$, $f_{b',N}$ has all of its roots real and simple.

(b) If l is odd and for all x in $(0, b^4)$, $f_{b,l}(-x) \geq 0$, then for all $\sqrt{3} < b' < b$, $f_{b',N}$ has nonreal roots.

Proof. (a) We show that $f_{b,N}$ has $N+1$ sign changes along the negative reals, implying the result. Write $N = 2r-1$ or $2r$, depending on whether N is odd or even. First, we will define a sequence $0 = x_0 < x_1 < x_2 < \cdots < x_{r-1}$ of r positive real numbers such that $\operatorname{sign}\left(f_{b,N}(-x_j)\right) = (-1)^j$, and then we will use the reparameterization to a symmetric polynomial, to show that this set can be extended to a strictly increasing sequence of $N+1$ positive real numbers with the same property.

To begin, we show that $f_{b,N}(-x_1) < 0$ (where $x_1 > 1$ satisfies the conditions in the statement of this lemma). The sequence of coefficients of $f_{b,N}$ is strongly unimodal, hence $\left\{x^j b^{-j(j+1)}\right\}$ is strongly unimodal and thus unimodal for any choice of $x > 0$. Set $q_j = x_1^j b^{-j(j+1)}$. Then

$q_{l+2}/q_{l+1} = x_1 b^{-2l+4}$; since $l \geq 2$, sufficient that this ratio be less than 1 is that $x_1 < b^8$, which is more than satisfied. Hence the sequence $\{q_j\}_{j \geq l+1}$ is monotone descending (since the whole sequence is unimodal), and since the first two terms are strictly decreasing, we see $\sum_{j \geq l+1}(-1)q^j$ has the same sign as $(-1)^{l+1}q_{l+1}$. This is -1 (as l is even). On the other hand, $f_{b,N}(-x_1) = f_{b,l}(-x_1) + \sum_{j \geq l+1}(-1)q^j$, which is thus negative.

For $k = 0$, set $x_0 = 0$, so that $f_{b,N}(x_0) = 1 = (-1)^0$. Now for each k with $2 \leq k \leq N - l$, define $x_k = x_1 b^{2k-2}$. Since $b > 1$, we have that $\{x_k\}_{k \geq 1}$ is strictly increasing, and since $N \geq 2l$, there are at least $N/2$ of them. Set $a(j) = j(j+1)$. Now we note the following self-replicating property (line 4 of the display below) of these polynomials. Set $x = Xb^\alpha$ where $\alpha = 2k - 2$

$$\begin{aligned}
f_{b,N}(x) &= f_{b,k-2}(x) + b^{-a(k-1)}x^{k-1}\sum_{j=0}^{l} x^j b^{a(k-1)-a(j+k-1)} + x^{k+l}\sum_{j=0} x^j b^{-a(j+k+l)} \\
&= f_{b,k-2}(x) + b^{-a(k-1)}x^{k-1}\sum_{j=0}^{l} X^j b^{j\alpha + a(k-1)-a(j+k-1)} + x^{k+l}\sum_{j=0} x^j b^{-a(j+k+l)} \\
&= f_{b,k-2}(x) + b^{-a(k-1)}x^{k-1}\sum_{j=0}^{l} X^j b^{-j(j+1-\alpha+2k-2)} + x^{k+l}\sum_{j=0} x^j b^{-a(j+k+l)} \\
&= f_{b,k-2}(x) + b^{-a(k-1)}x^{k-1} f_{b,l}(X) + x^{k+l}\sum_{j=0} x^j b^{-a(j+k+l)} \\
&= f_{b,k-2}(x) + b^{-a(k-1)}x^{k-1} f_{b,l}(xb^{2-2k}) + x^{k+l}\sum_{j=0} x^j b^{-a(j+k+l)}; \quad \text{since } l \text{ is even,} \\
f_{b,N}(x) &= f_{b,k-2}(-x) + (-1)^{k-1}C_k f_{b,l}(-xb^{2-2k}) + (-1)^k x^{k+l}\sum_{j=0}(-x)^j b^{-a(j+k+l)},
\end{aligned}$$

where $C_k > 0$. Substitute $x = x_k = x_1 b^{2k-2}$. The middle term is then $(-1)^{k-1}C_k f_{b,l}(-x_1)$; by hypothesis, this is either zero or has sign $(-1)^k$. For the two tails (left and right; at least one of them must be nonempty), set $q_j = (x_k)^j b^{-a(j)}$. Again, since $\left\{b^{-a(j)}\right\}_{j=0}^{N}$ is strongly unimodal, so is the reparameterized sequence $\{q_j\}$. If $k = 2$, the left tail consists of a single term 1, which has sign $(-1)^k$. If $k \geq 3$, the left tail sums to $\sum_{j=0}^{k-2}(-1)^j q_j$. We note that $q_{k-2}/q_{k-3} = x_k b^{6-2k} = x_1 b^4 > 1$. Hence the sequence $\{q_j\}_{j=0}^{k-2}$ is increasing, with the last difference strict. Hence sign $\left(\sum_{j=0}^{k-2}(-1)^j q_j\right)$ is the sign of the largest term, $(-1)^{k-2}q_{k-1}$, i.e., $(-1)^k$.

The right tail is treated similarly. If $k + l = N$, there is only one term, and its sign is $(-1)^{k+l} = (-1)^k$; otherwise, suppose $k + l < N$. We note that $q_{k+l}/q_{k+l+1} = x_k^{-1} b^{2(k+l+1)} = x_1^{-1} b^{2l} \geq b^4/x_1 > 1$. As an interval in a strongly unimodal sequence, it follows that $\{q_j\}_{j \geq k+l}$ is descending, and thus the sign of $\sum_{j \geq k+l}(-1)^j q_j$ is that of the initial term, i.e., $(-1)^{k+l} = (-1)^k$.

Hence each of the three parts is either zero or has sign that of $(-1)^k$, and at least one of the three parts is not zero. Hence sign $(f_{b,N}(-x_k)) = (-1)^k$.

The k for which this is valid include all k with $k \leq N - l$, and since $N \geq 2l$ by hypothesis, this is true for all $k \leq N/2$. Now consider the symmetric form of $f_{b,N}$, given above as $\mathcal{F}_{b,N}$ where $\mathcal{F}_{b,N}(x) = f_{b,N}(xb^{N+1})$. For $k \leq N/2$, set $X_k = x_k b^{-(N+1)}$, so that $\mathcal{F}_{b,N}(-X_k) = f_{b,N}(-x_k)$. Hence sign $(\mathcal{F}_{b,N}(-X_k)) = (-1)^k$. Next, we note that $0 = X_0 < X_1 < X_2 < \ldots$; moreover, $X_k = x_1 b^{2k-2-N-1}$. Hence if $2k < N$, i.e., $2k \leq N-1$, then $X_k < 1$ (as $x_1 < b^4$). Now we consider the two cases, N even and N odd.

If $N = 2r$ is even, then we have $0 = X_0 < \cdots < X_{r-1} < 1$; set $X_r = 1$. Then $\mathcal{F}_{b,N}(1)$ is up to positive scalar multiple, sign $((-x)^r))\,(1-2/b+2/b^4-\ldots)$. Now $1-2/b+2/b^4-2/b^9 > 0$ if $b > 1.44$,

and since we have assumed $b > \sqrt{3} \equiv 1.73\ldots$, it follows easily that $\text{sign}\,(\mathcal{F}_{b,N}(1)) = (-1)^r$. Now for $r+1 \le k \le N-1 = 2r-1$, set $X_k = (X_{2r-k})^{-1}$. Then $X_j < X_{j+1}$ for all $0 \le j \le N-2$, and for $k > r$, we have $\text{sign}\,(\mathcal{F}_{b,N}(-X_k)) = (-1)^N \text{sign}\,(\mathcal{F}_{b,N}(-X_{2r-k})) = (-1)^{2r-k} = (-1)^k$. Finally, there exists sufficiently large $X' > X_{2r-1}$ such that $\text{sign}\,(\mathcal{F}_{b,N}(X)) = (-1)^N = 1$; set $X_N = X'$. We thus have $N+1$ sign changes in the values of $\text{sign}\,(\mathcal{F}_{b,N})$ on the negative real numbers, hence $\mathcal{F}_{b,N}$ has at least N distinct real roots, and thus these must exhaust them. Since $f_{b,N}$ is a reparameterization of $\mathcal{F}_{b,N}(1)$, the same applies to $f_{b,N}$.

If $N = 2r-1$ is odd, then we have $0 = X_0 < X_1 < \ldots X_{r-1} < 1$. For $r \le k \le 2r-2$, set $X_k = X_{N-k}^{-1}$, and define X_N to be a sufficiently large number that $X_N > X_{2r-2}$ and $\text{sign}\,(\mathcal{F}_{b,N}(-X_N)) = (-1)^N$. Then we have $X_j < X_{j+1}$ for $0 \le j \le N-1$, and moreover, for $N-1 \ge k \ge r$, we have $\text{sign}\,(\mathcal{F}_{b,N}(-X_k)) = (-1)^N \text{sign}\,(\mathcal{F}_{b,N}(-X_{N-k})) = (-1)^{N+N-k} = (-1)^k$. Thus again $\mathcal{F}_{b,N}$ has $N+1$ sign changes on the negative reals, so $\mathcal{F}_{b,N}$, and therefore $f_{b,N}$ has N distinct negative roots.

Now define (for fixed l) a function of two variables $G(b,Y)$, via $G(b,Y) = f(Yb^4)$, so that

$$
\begin{aligned}
G(b,Y) &= 1 + b^2Y + b^2Y^2 + Y^3 + Y^4b^{-4} + \ldots \\
&= 1 + b^2Y + b^2Y^2 + Y^3 + \sum_{j=4}^{l} \frac{Y^j}{b^{j(j+1)-4j}} \\
\frac{\partial G}{\partial b}(b,Y) &= 2(Y^2+Y)b - \sum_{j=2}^{l/2-1} \frac{Y^{2j}}{b^{4j^2-6j+1}} \left(4j^2 - 6j + \frac{((2j+1)(2j+2) - 8j - 4)Y}{b^{4j}}\right) \\
&\quad - \begin{cases} Y^l C_l & \text{if } l \text{ is even; } C_l > 0 \\ \frac{Y^{l-1}}{l(l-1)-4l+5}\left((l-1)l - 4j + 4 - \frac{(l(l+1)-4l)Y}{b^{2l}}\right) & \text{if } l \text{ is odd} \end{cases}
\end{aligned}
$$

It follows easily that if $l \ge 2$, $0 < Y < 1$, and $\sqrt{3} < b < 2$, then $\frac{\partial G}{\partial b}(b,-Y) < 0$.

Suppose that for some $\sqrt{3} \le b_0 < 2$ and $0 < x_0 < b_0^4$, we have $f_{b_0,l}(-x_0) \le 0$. Then $G(b_0, -xb_0^{-4}) = 0$ and $0 < xb_0^{-4} < 1$. By the previous paragraph, it follows that for all $\sqrt{2} > b > b_0$, we have $G(b, x_0b_0^{-4}) < 0$ and therefore $f_{b,l}(x_0b^4/b_0^4) \le 0$. We have thus shown that if $f_{b_0,l}$ hits zero or less on the interval $(-b_0^4, 0)$ and $\sqrt{3} < b_0 < 2$, then if $2 > b > b_0$, the function $f_{b,l}$ hits zero or less on the interval $(-b^4, 0)$.

In particular, if l is even, $N \ge 2l$, $\sqrt{3} < b_0 < 2$, and $f_{b_0,l}$ hits zero or less on the interval $(-b_0^4, 0)$, then for all b with $2 \ge b \ge b_0$, the function $f_{b,N}$ has only real and simple zeros.

(b) Suppose now that l is odd, and for all x with $0 < x < b^4$ where $b > \sqrt{3}$, $f_{b,l}(-x) \ge 0$. We show that this implies $f_{b,N}$ has nonreal roots (when $N \ge 2l$), and the same is true when b is decreased.

We show that $f'_{b,N}$ (the derivative is with respect to x) has a zero in $(-b^4, 0)$, and that $f_{b,N}$ is strictly positive on $[-b^4, 0]$. If $f_{b,N}$ had only real zeros, this yields a contradiction, since in that case, the zeros of f are intertwined by those of f'.

We write

$$
\begin{aligned}
f_{b,N}(-x) &= f_{b,l}(-x) + \sum_{j=1}^{N-l} \frac{(-x)^{l+j}}{b^{(l+j)(l+j+1)}} \\
&= f_{b,l}(-x) + \frac{x^l}{b^{l^2+l}} \sum_{j=1}^{N-l} \frac{(-1)^{j+1}x^j}{b^{(j)(2l+j+1)}}
\end{aligned}
$$

The series is alternating and its first term ($j=1$) is positive. We check that the series is monotone

decreasing in absolute value; this boils down to

$$\frac{x^j}{b^{j(j+2l+1)}} > \frac{x^{j+1}}{b^{(j+1)(j+2l+2)}}; \qquad \text{that is,}$$
$$x < b^{2j+2+l},$$

for which $x < b^4$ is more than sufficient. Hence the alternating series is at least as large as the sum of its first two terms, which is positive. Since $f_{b,l}(-x) \geq 0$ by hypothesis, we have that $f_{b,N}(-x) > 0$.

Next, we see that $f'_{b,N}(0) = 1/b^2 > 0$ and

$$\begin{aligned} f'_{b,N}(-b^4) &= \sum_{j=1}^{N} (-1)^{j-1} j b^{-(j^2-3j+4)} \\ &= -b^{-2} + 3b^{-4} - 4b^{-8} + 5b^{-14} \\ &< 0 \end{aligned}$$

as $b^2 > 3$. Hence f' has a zero in the interval $(-b^4, 0)$. Since $f_{b,N}$ is strictly positive on $[-b^4, 0]$, it easily follows that $f_{b,N}$ has nonreal roots.

Next, suppose that $f_{b_0,N}$ is strictly positive on $[-b_0^4, 0]$ and $b < b_0$. We show that $f_{b,N}$ is strictly positive on $[-b^4, 0]$. With the same G as defined previously, since $\frac{\partial G}{\partial b}(b, -Y) < 0$ on the relevant interval, the result is immediate. •

The optimal procedure is to look for those values of $b \sim b_0$ in the interval $(\sqrt{3}, 2)$ for which the polynomial (in x) $f_{b,l}$ has a multiple zero at a point in $(0, b^4)$. The even values of l give lower bounds, the odd values give upper bounds. We find convergence is extremely fast—by $l = 11$, we are within 10^{-24} of B_0, the critical value: if $b > B_0$, then $f_{b,N}$ has all of its seros real and simple for all sufficiently large N (including $N = \infty$), and if $b < B_0$, then $f_{b,N}$ has nonreal zeros for all sufficiently large N. If $N = \infty$, it follows that $f_{B_0,N}$ has only real zeros (possibly with multiplicities), but if $N < \infty$, then it necessarily has nonreal zeros. It turns out that B_0 is just less than $\sqrt{1+\sqrt{5}}$. Remember that the "β" value (the constant ratio $c_j^2/c_{j+1}c_{j-1}$) is the *square* of b.

To check for multiple roots, we use the discriminant and *Maple*.

First for odd l; for $l = 3$, $b_0 = \sqrt{3}$, i.e., $f_{b_0,3}$ has a multiple (in fact, a triple) zero in the interval $(0, 9)$. This would yield that if $b < \sqrt{3}$, then $f_{b,N}$ has nonreal roots for $N \geq 7$ if we extended the proposition to cover the endpoint (which is a nuisance). It is easy to show this anyway (keep in mind however that $(1+x)^3$ is a reparameterization of $f_{\sqrt{3},3}$, and it has real zeros, albeit multiple). This computation can be done by hand.

The remaining values were obtained by *Maple,* using 50-digit accuracy (truncated to 25 digits here):

$$\begin{aligned} l &= 5 & b_0 &= 1.798227032486330299597020 1 \\ l &= 7 & b_0 &= 1.7982315382687507032044628 \\ l &= 9 & b_0 &= 1.7982315382745004887263767 \\ l &= 11 & b_0 &= 1.7982315382745004887933797 \end{aligned}$$

Now for even l; with $l = 2$, we obtain $b_0 = 2$, which yields nothing we didn't already know, namely that if all the ratios are four or more, all the roots are real and simple. However, $l = 4$ yields $b_0 = \sqrt{1+\sqrt{5}}$ (computable by hand), which means that if $b^2 \geq 1+\sqrt{5} \sim (1.79891\ldots)^2$

(3.236...), then $f_{b,N}$ has only real and simple zeros for $N \geq 8$. Notice how close this number is to the lower bound obtained from $l = 5$ (they differ at the fourth place). The rest of the values were computed by *Maple*, with the same constraints as above. Here we also have to verify that x_0, one the values of x where the multiple root occurs, can be chosen in the interval $(0, b_0^4)$. Since $b_0^4 > 10$ (from the lower bounds obtained in the odd cases), we only need a rough approximation to the values of x_0.

When $l = 4$, $f_{(1+\sqrt{5})^{1/2},4}(x)$ is a quartic and a square, whose roots are

$$\left\{-14 - 65^{1/2} + 2(50 + 225^{1/2})^{1/2}, -14 - 65^{1/2} - 2(50 + 225^{1/2})^{1/2}\right\},$$

each with multiplicity two. The relevant double zero is the first one, so $x_0 = -14 - 65^{1/2} - 2(50 + 225^{1/2})^{1/2}$, approximately 7.49722... (remember that x_0 is the negative of the zero of the polynomial). This is well within the upper bound of $(1 + \sqrt{5})^2$. For $l = 6$, the discriminant of $f_{b,6}$ is up to multiplication by b^{-210}, an even polynomial in b of degree 70, but even so, we obtain $x_0 \sim 7.503\ldots$, and for $l = 8$ and 10, the corresponding values of x_0 are the same to within six decimals.

$$\begin{aligned}
l &= 4 \qquad b_0 = 1.7989074399478672722612275\\
l &= 6 \qquad b_0 = 1.7982315474312892803918067\\
l &= 8 \qquad b_0 = 1.7982315382745016049847445\\
l &= 10 \qquad b_0 = 1.7982315382745004887933809
\end{aligned}$$

Now B_0 is squeezed between the supremum of the values of b_0 for odd l and the infimum of the values of b_0 for even l. Taking the numbers from $l = 10$ and $l = 11$, we have

$$|B_0 - 1.7982315382745004887933803| < 6 \times 10^{-25}$$

Maple (and consequently, I) gave up at $l = 12$, but presumably it would have yielded accuracy of order 10^{-31}. In any case, $B_0^2 \sim 3.23364$ is likely good enough.

As an aside, the self-replicating property discussed above for the polynomials is more clearly seen in the functional equation satisfied by the entire function $f_{b,\infty}$, specifically,

$$f(x) = 1 + \frac{x}{b^2} f\left(\frac{x}{b^2}\right),$$

which can be iterated to more confusing forms.

An interesting phenomenon can be observed. From section 1, we can find polynomials of all sufficiently large degrees or an entire function for which all the ratios $c_j^2/c_{j+1}c_{j-1}$ exceed 3.99, yet which has nonreal zeros. On the other hand, if all these ratios are equal to 3.24 (and the degree is large enough), all the zeros are real. This seems counter-intuitive—the larger the ratios, the better behaved we expect the polynomial to be with respect to its zeros. However, if we examine the arguments in section 1, we see that when the ratios are large, the coefficients tail off very quickly, and so only contiguous islands of coefficients with the same signs (see the argument there) can be used.

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Mathematics Dept, University of Ottawa, Ottawa ON K1N 6N5, Canada; e-mail: dehsg@uottawa.ca